

# CONTINUED $g$ -FRACTIONS AND GEOMETRY OF BOUNDED ANALYTIC MAPS

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**ABSTRACT.** In this work we study qualitative properties of real analytic bounded maps. The main tool is approximation of real valued functions analytic in rectangular domains of the complex plane by continued  $g$ -fractions of Wall [8]. As an application, the Sundman-Poincaré method in the Newtonian three-body problem is revisited and applications to collision detection problem are considered.

## 1. THE CONTINUED $g$ -FRACTION REPRESENTATION FOR REAL ANALYTIC BOUNDED FUNCTIONS

By  $R_{T,B} \subset \mathbb{C}$  we denote the open domain which is the interior of the rectangle with vertices at the points  $T + iB$ ,  $T - iB$ ,  $-T + iB$ ,  $-T - iB$ ,  $T$ ,  $B > 0$  (see Fig. 1). Let  $\mathbb{A}_{M,T,B}$  be the set of all functions  $f(z)$  analytic in  $R_{T,B}$ , real valued for  $z \in I_T = (-T, T)$  and bounded in absolute value in  $R_{T,B}$  by  $M > 0$ :

$$|f(z)| < M, \quad \forall z \in R_{T,B}. \quad (1.1)$$

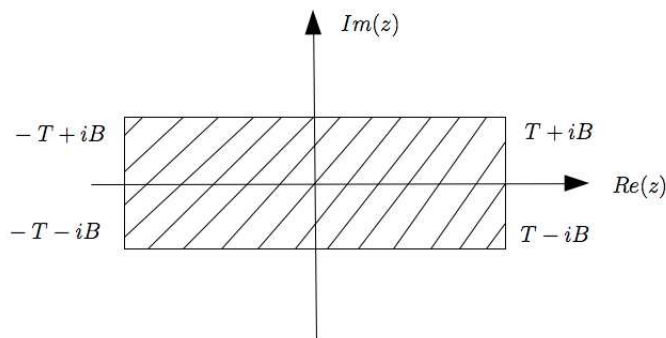


FIGURE 1. Domain of analiticity  $R_{T,B}$  of  $f(z) \in \mathbb{A}_{M,T,B}$ .

Let  $\mathbb{H} = \mathbb{C}_- \cup \mathbb{C}_+ \cup (-1, +\infty)$  where  $\mathbb{C}_+ = \{z \in \mathbb{C} : \text{Im}(z) > 0\}$ ,  $\mathbb{C}_- = \{z \in \mathbb{C} : \text{Im}(z) < 0\}$ . We shall construct the conformal map between the open connected sets  $R_{T,B}$  and  $\mathbb{H}$  using the Jacobi elliptic function  $\text{sn}(z, k)$  and the theta functions  $\theta_2(z, q)$ ,  $\theta_3(z, q)$  ( see for definitions [1] ).

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Let

$$q = e^{-\frac{\pi B}{T}}, \quad (1.2)$$

be the nome.

The corresponding real quarter-period  $K > 0$  is defined as follows

$$K = \frac{\pi}{2} \theta_3(0, q)^2. \quad (1.3)$$

The elliptic modulus  $k \in (0, 1)$  is given by formula

$$k = \frac{\theta_2(0, q)^2}{\theta_3(0, q)^2}, \quad (1.4)$$

and defines the Jacobi elliptic function  $\text{sn}(z, k)$ .

**Lemma 1.1.** *Let*

$$\Phi(z) = \frac{2\text{sn}(\frac{Kz}{T}, k)}{1 - \text{sn}(\frac{Kz}{T}, k)}. \quad (1.5)$$

*Then  $\Phi : R_{T,B} \rightarrow \mathbb{H}$  is conformal and maps bijectively  $(-T, T)$  to  $(-1, +\infty)$ ,  $\Phi(0) = 0$ .*

The proof is straightforward and easily follows from properties of  $\text{sn}(z, k)$  described in [2], p. 119. Let

$$\mathbb{D}_M = \{z \in \mathbb{C} : |z| < M\}, \quad \mathbb{H}_+ = \{z \in \mathbb{C} : \text{Re}(z) > 0\}. \quad (1.6)$$

One verifies that  $m : \mathbb{D}_M \rightarrow \mathbb{H}_+$  defined by

$$m(z) = \frac{M + z}{M - z}, \quad (1.7)$$

is conformal.

The composition

$$F = m \circ f \circ \Phi^{-1}, \quad \text{where } f \in \mathbb{A}_{M,T,B}, \quad (1.8)$$

is then holomorphic function in  $\mathbb{H}$  such that  $F(\mathbb{H}) \subset \mathbb{H}_+$  and  $F : (-1, +\infty) \rightarrow (0, +\infty)$ .

According to Theorem of Wall [8], p. 279 there exist  $\mu_0 > 0$  and the sequence of real numbers

$$g_i \in [0, 1], \quad i \geq 1, \quad (1.9)$$

such that

$$F(z) = \mu_0 \sqrt{1 + z} \{g_1, g_2, \dots | z\}, \quad z \in \mathbb{H}, \quad (1.10)$$

where

$$g(z) = \{g_1, g_2, \dots | z\} = \frac{1}{1} + \frac{g_1 z}{1} + \frac{(1 - g_1)g_2 z}{1} + \frac{(1 - g_2)g_3 z}{1} \dots, \quad (1.11)$$

is a continued  $g$ -fraction converging uniformly on compact sets of  $\mathbb{H}$  to an analytic function  $g(z)$ ,  $z \in \mathbb{H}$ .

**Remark 1.1.**  $g(z)$  is a rational function of  $z$  if and only if  $g_k \in \{0, 1\}$ , for some  $k \geq 1$ .

As follows from (1.8):  $f = m^{-1} \circ F \circ \Phi$  and hence the following representation for  $f(z)$  holds

$$f(z) = M \left( 1 - \frac{2}{\mu_0 \sqrt{1 + \Phi(z)} \{g_1, g_2, \dots | \Phi(z)\} + 1} \right), \quad z \in R_{T,B}. \quad (1.12)$$

To simplify (1.12) we make the rescaling and obtain the new function  $\phi$  given by

$$\phi(z) = \frac{f(z/\alpha)}{M}, \quad \alpha = \frac{K}{T}, \quad (1.13)$$

which is holomorphic in the rectangle  $R_{K,\alpha B}$ .

We note that  $|\phi(z)| < 1$ ,  $\forall z \in R_{K,\alpha B}$ .

Formula (1.12) then becomes

$$\phi(z) = \left( 1 - \frac{2}{\mu_0 \sqrt{1 + \eta(z)} \{g_1, g_2, \dots | \eta(z)\} + 1} \right), \quad z \in R_{K,\alpha B}, \quad (1.14)$$

where

$$\eta(z) = \frac{2\operatorname{sn}(z, k)}{1 - \operatorname{sn}(z, k)}. \quad (1.15)$$

The map  $z \mapsto \eta(z)$  is a bijection between the real intervals  $(-K, K)$  and  $(-1, +\infty)$ ,  $\eta(0) = 0$  what will be used later.

We define the truncated continued  $g$ -fraction as the  $n$ -order approximation of (1.11):

$$\{g_1, g_2, \dots, g_n | z\} = \frac{1}{1} + \frac{g_1 z}{1} + \frac{(1 - g_1)g_2 z}{1} \dots \frac{(1 - g_{n-1})g_n z}{1}, \quad (1.16)$$

which is a rational function of  $z$  analytic in  $\mathbb{H}$ .

The next theorem gives the *a priori* bounds for the  $g$ -fraction (1.11).

**Theorem 1.1.** ([5])

a) Let  $k = 2n + 1$ ,  $n = 0, 1, \dots$ , then

$$A_k(z) \leq g(z) \leq B_k(z), \quad -1 < z < +\infty, \quad (1.17)$$

where

$$A_k(z) = \{g_1, g_2, \dots, g_k | z\}, \quad B_k(z) = \{g_1, g_2, \dots, g_k, 1 | z\}. \quad (1.18)$$

b) Let  $k = 2n$ ,  $n = 1, 2, \dots$ , then

$$A_k^+(z) \leq g(z) \leq B_k^+(z), \quad 0 \leq z < +\infty, \quad (1.19)$$

$$A_k^-(z) \leq g(z) \leq B_k^-(z), \quad -1 < z < 0, \quad (1.20)$$

where

$$A_k^+(z) = \{g_1, g_2, \dots, g_k, 1 | z\}, \quad B_k^+ = \{g_1, g_2, \dots, g_k | z\}, \quad (1.21)$$

and  $A_k^- = B_k^+$ ,  $B_k^- = A_k^+$ .

Using the above formulas we write below the rational *a priori* bounds for the  $g$ -fraction (1.11) corresponding to  $k = 1, 2, 3$ :

Case  $k = 1$ .

$$A_1(z) = \frac{1}{1 + g_1 z}, \quad B_1(z) = \frac{1 + (1 - g_1)z}{1 + z}. \quad (1.22)$$

Case  $k = 2$ .

$$A_2^+(z) = \frac{(1 - g_1 g_2)z + 1}{(1 + z)(g_1(1 - g_2)z + 1)}, \quad B_2^+(z) = \frac{g_2(1 - g_1)z + 1}{(g_1 - g_1 g_2 + g_2)z + 1}, \quad (1.23)$$

$$A_2^- = B_2^+, B_2^- = A_2^+. \quad (1.24)$$

Case  $k = 3$ .

$$A_3(z) = \frac{(g_3 + g_2 - g_3 g_2 - g_2 g_1)z + 1}{g_1 g_3(1 - g_2)z^2 + (g_3 + g_2 + g_1 - g_3 g_2 - g_1 g_2)z + 1}. \quad (1.25)$$

$$B_3(z) = \frac{g_2(1 - g_3)(1 - g_1)z^2 + (1 + g_2 - g_3 g_2 - g_1 g_2)z + 1}{(1 + z)((g_1 + g_2 - g_3 g_2 - g_1 g_2)z + 1)}. \quad (1.26)$$

The coefficients  $g_p$  in formula (1.14) are defined by

$$g_p = R_p(\phi(0), \phi'(0), \dots, \phi^{(p)}(0)), \quad p \geq 1, \quad (1.27)$$

with rational expressions  $R_p$  which can be found by calculation of derivatives of both sides of (1.14) and evaluating them at  $z = 0$ . The recurrent formulas for  $R_p$  can be derived from [8], p. 203.

In view of (1.13),  $\phi_n$  are functions of derivatives  $f^{(n)}(0)$ :

$$\phi_n = \frac{\alpha^{-n}}{M} f^{(n)}(0), \quad \alpha = K/T, \quad n \geq 0. \quad (1.28)$$

Below we give explicit expressions for  $\mu_0, g_1, g_2$  in terms of  $\phi_n, n = 0, 1, 2$ :

$$\mu_0 = \frac{1 + \phi_0}{1 - \phi_0}, \quad (1.29)$$

$$g_1 = \frac{1}{2} \frac{1 - \phi_0^2 - 2\phi_1}{1 - \phi_0^2}, \quad (1.30)$$

$$g_2 = \frac{1}{2} \frac{(4\phi_1^2 - 2\phi_2 - \phi_0 + \phi_0^2 + \phi_0^3 - 2\phi_2\phi_0 - 1)(1 - \phi_0)}{(2\phi_1 - \phi_0^2 + 1)(2\phi_1 + \phi_0^2 - 1)}. \quad (1.31)$$

**Corollary 1.1.** *As seen from (1.30),  $f'(0) > 0$  is equivalent to  $g_1 < 1/2$ ;  $f'(0) < 0$  is equivalent to  $g_1 > 1/2$  and  $f'(0) = 0 \Leftrightarrow g_1 = 1/2$ .*

## 2. BOUNDS ON THE TIME OF THE FIRST RETURN

Applying Theorem 1.1 to the  $g$ -fraction in (1.12) one can derive the *a priori* bounds for  $f(z)$  holding inside the interval  $(-T, T)$ . Increasing the truncation order  $n$  in (1.16) one obtains more and more precise information of qualitative character about  $f(z)$  once the derivatives of  $f(z)$  at  $z = 0$  are known. In particular, if  $f(z)$  is a solution of a system of analytic differential equations, one can find often recurrent formulas to calculate derivatives  $f^{(n)}(0)$  of all orders  $n \geq 0$  and write the  $g$ -fraction representation (1.12).

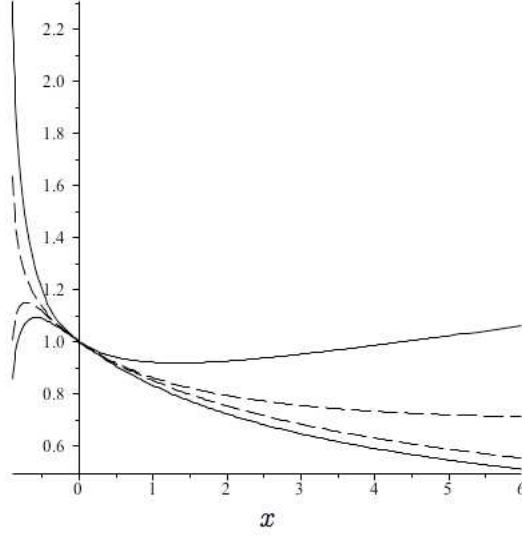


FIGURE 2. Bounds  $r(x)A_1(x)$ ,  $r(x)B_1(x)$  (bold line) and  $r(x)A_2^\pm(x)$ ,  $r(x)B_2^\pm(x)$  (dashed line) for  $x \in (-0.9, 6)$ ,  $g_1 = 0.7$ ,  $g_2 = 0.3$ ,  $r(x) = \sqrt{1+x}$ .

Our aim is to estimate the time of return of  $f(z)$  to the initial value  $f(0)$  i.e to study the points  $z_0 \in (-T, T)$ ,  $z_0 \neq 0$  such that  $f(z_0) = f(0)$ . To do this we will use the *a priori* bounds (1.22) applied to the  $g$ -fraction in formula (1.14). For  $p = 2k + 1$  one obtains:

$$\left(1 - \frac{2}{\mu_0 \sqrt{1 + \eta(z)} A_p(\eta(z)) + 1}\right) \leq \phi(z) \leq \left(1 - \frac{2}{\mu_0 \sqrt{1 + \eta(z)} B_p(\eta(z)) + 1}\right), \quad (2.1)$$

for  $z \in (-K, K)$ .

If  $p = 2k$  then

$$\left(1 - \frac{2}{\mu_0 \sqrt{1 + \eta(z)} A_p^+(\eta(z)) + 1}\right) \leq \phi(z) \leq \left(1 - \frac{2}{\mu_0 \sqrt{1 + \eta(z)} B_p^+(\eta(z)) + 1}\right), \quad (2.2)$$

for  $z \in (0, K)$ , and

$$\left(1 - \frac{2}{\mu_0 \sqrt{1 + \eta(z) A_p^-(\eta(z)) + 1}}\right) \leq \phi(z) \leq \left(1 - \frac{2}{\mu_0 \sqrt{1 + \eta(z) B_p^-(\eta(z)) + 1}}\right), \quad (2.3)$$

for  $z \in (-K, 0]$ .

**Definition 2.1.** We denote by  $\mathbb{A}_{M,T,B}^{(k)} \subset \mathbb{A}_{M,T,B}$ ,  $k = 1, 2, \dots$  the set of functions for which the  $g$ -fraction representation (1.12) satisfies the condition

$$g_i \notin \{0, 1\}, \quad \forall i = 1, \dots, k. \quad (2.4)$$

In the next theorem, for a given  $f \in \mathbb{A}_{M,T,B}^{(1)}$ , we will describe a neighborhood of origin in which  $z = 0$  is the only solution of  $f(z) = f(0)$ .

**Theorem 2.1.** Let  $f(z) \in \mathbb{A}_{T,B,M}^{(1)}$ ,  $f'(0) \neq 0$  where  $g_1$  is defined by (1.2), (1.3), (1.28) and (1.30) as a function of  $M, T, B$ ,  $f(0)$ ,  $f'(0)$ . Let  $\tau \in (-T, T)$ ,  $\tau \neq 0$  be the point such that  $f(\tau) = f(0)$ . We define

$$\tilde{\tau} = \frac{T}{K} |\operatorname{sn}^{-1}(g, k)| = \frac{T}{K} \left| \int_0^g \frac{dt}{\sqrt{1-t^2} \sqrt{1-k^2 t^2}} \right|, \quad (2.5)$$

where

$$g = \frac{1 - 2g_1}{g_1^2 + (1 - g_1)^2} \in (-1, 1), \quad (2.6)$$

and  $k$ ,  $K$  are given by (1.4) and (1.3).

Then  $0 < \tilde{\tau} < T$  and

$$|\tau| \geq \tilde{\tau}. \quad (2.7)$$

*Proof.* One considers (2.1) with  $p = 1$ . We have  $A_1(0) = B_1(0) = 1$  and define  $t_1, t_2$  as non-zero solutions of the following algebraic equations

$$\sqrt{1+t_1} A_1(t_1) = 1, \quad \sqrt{1+t_2} B_1(t_2) = 1, \quad t_1, t_2 \in (-1, +\infty). \quad (2.8)$$

Simple algebraic calculations show that the only solutions satisfying (2.8) are given by

$$t_1 = \frac{1 - 2g_1}{g_1^2}, \quad t_2 = \frac{2g_1 - 1}{(1 - g_1)^2}, \quad (2.9)$$

which are related by

$$\frac{1}{t_1} + \frac{1}{t_2} = -1. \quad (2.10)$$

Since  $\eta(z)$  is a bijection between the intervals  $(-K, K)$  and  $(-1, +\infty)$  there exist unique real numbers  $T_1, T_2 \in (-K, K)$  satisfying the following equations:

$$\eta(T_1) = t_1, \quad \eta(T_2) = t_2. \quad (2.11)$$

As easily seen from (1.15):  $T_1 = -T_2$  and

$$\eta^{-1}(y) = \int_0^{\frac{y}{2+y}} \frac{dt}{\sqrt{1-t^2} \sqrt{1-k^2 t^2}}, \quad y \in (-1, 1). \quad (2.12)$$

We define

$$\tilde{\tau} = \frac{T}{K}|T_1| = \frac{T}{K}|\eta^{-1}(t_1)| \in (0, T). \quad (2.13)$$

The proof of Theorem 2.1 follows therefore directly from (2.1) and (2.8).  $\square$

The next result shows that  $f(z) \in \mathbb{A}_{T,B,M}^{(2)}$ , under some conditions on derivatives  $f^{(p)}(0)$ ,  $p = 0, 1, 2$ , always returns to the initial value  $f(0)$  in the interval  $(-T, T)$  i.e admits the oscillatory property.

**Theorem 2.2.** *Let  $f(z) \in \mathbb{A}_{T,B,M}^{(2)}$ ,  $f'(0) \neq 0$  where  $g_1, g_2$  are defined by formulas (1.2), (1.3), (1.28) and (1.30), (1.31). We assume that one of the two following conditions (A) or (B) holds*

$$g_1 < 1/2, \quad g_1^2 - 4(1 - g_1)^2(1 - g_2)g_2 \geq 0 \quad (A) \quad (2.14)$$

$$g_1 > 1/2, \quad (1 - g_1)^2 - 4g_1^2g_2(1 - g_2) \geq 0 \quad (B) \quad (2.15)$$

Then there exists  $\tau \in (-T, T)$ ,  $\tau \neq 0$  such that

$$f(\tau) = f(0). \quad (2.16)$$

*Proof.* We consider (2.2) with  $p = 2$  and define the following real algebraic equations

$$\sqrt{1+x} A_2^+(x) = 1, \quad x \in (0, +\infty), \quad (A_1),$$

$$\sqrt{1+x} B_2^+(x) = 1, \quad x \in (0, +\infty), \quad (B_1),$$

$$\sqrt{1+x} B_2^-(x) = 1, \quad x \in (-1, 0), \quad (\tilde{A}_1),$$

$$\sqrt{1+x} A_2^-(x) = 1, \quad x \in (-1, 0). \quad (\tilde{B}_1),$$

where  $A_2^- = B_2^+$ ,  $B_2^- = A_2^+$ .

Making the change of variables

$$x = -1 + t^2, \quad t \in \mathbb{R}, \quad (2.17)$$

after some elementary transformations, it is easy to show that equations  $(A_1)$ ,  $(B_1)$  are equivalent respectively to quadratic equations  $(A_2)$  and  $(B_2)$  given below

$$P_1(t) = g_1(1 - g_2)t^2 - (1 - g_1)t + g_1g_2 = 0, \quad t \in \mathbb{R}, \quad (A_2)$$

$$P_2(t) = g_2(1 - g_1)t^2 - g_1t + (1 - g_1)(1 - g_2) = 0, \quad t \in \mathbb{R}. \quad (B_2)$$

**Remark 2.1.** We notice that  $P_2(t)$  is obtained from  $P_1(t)$  by transformation

$$g_i \mapsto 1 - g_i, \quad i = 1, 2. \quad (2.18)$$

The polynomial  $P_1(t) = 0$  has two real roots  $t_1^{(1)}, t_2^{(1)} \in \mathbb{R}$

$$t_1^{(1)} = \frac{1 - g_1 - \sqrt{D_1}}{2g_1(1 - g_2)}, \quad t_2^{(1)} = \frac{1 - g_1 + \sqrt{D_1}}{2g_1(1 - g_2)}, \quad t_1^{(1)} \leq t_2^{(1)}, \quad (2.19)$$

if and only if the following condition holds

$$D_1 = (1 - g_1)^2 - 4g_1^2g_2(1 - g_2) \geq 0. \quad (2.20)$$

$P_2(t) = 0$  has two real solutions  $t_1^{(2)}, t_2^{(2)} \in \mathbb{R}$

$$t_1^{(2)} = \frac{g_1 - \sqrt{D_2}}{2(1 - g_1)g_2}, \quad t_2^{(2)} = \frac{g_1 + \sqrt{D_2}}{2(1 - g_1)g_2}, \quad t_1^{(2)} \leq t_2^{(2)}, \quad (2.21)$$

if and only if

$$D_2 = g_1^2 - 4(1 - g_1)^2(1 - g_2)g_2 \geq 0. \quad (2.22)$$

Applying the Vieta's formulas to polynomials  $A_2$  and  $B_2$ , and taking into account that  $g_i \in (0, 1)$ ,  $i = 1, 2$  one checks that:

$$t_j^{(i)} > 0, \quad i, j = 1, 2. \quad (2.23)$$

*Case A.* Let  $f'(0) > 0 (\Leftrightarrow g_1 < 1/2)$ . Then  $f(z)$  is increasing function in the interval  $(-\epsilon, \epsilon)$  for some small  $\epsilon > 0$ . We assume that inequality  $D_2 \geq 0$  holds, so both roots  $t_1^{(2)}$  and  $t_2^{(2)}$  are real. One has  $P_2(1) = 1 - 2g_1 > 0$ , so, in view of (2.23), either  $0 < t_1^{(2)} \leq t_2^{(2)} < 1$  (a) or  $1 < t_1^{(2)} \leq t_2^{(2)}$  (b). One verifies with help of (2.21) that (a) is equivalent to  $L_2 = g_1 - 2(1 - g_1)g_2 < 0$  and (b) to  $L_2 > 0$ . Thus, in view of (2.17), if (b) holds, the equation  $(B_1)$  will have solution  $x = -1 + t_1^{(2)2} \in (0, +\infty)$  and if (a) holds,  $(\tilde{B}_1)$  will have solution  $x = -1 + t_2^{(2)2} \in (-1, 0)$ .

*Case B.* Let  $f'(0) < 0 (\Leftrightarrow g_1 > 1/2)$ . Then  $f(z)$  is decreasing function in the interval  $(-\epsilon, \epsilon)$  for some small  $\epsilon > 0$ . We assume that inequality  $D_1 \geq 0$  holds, so both roots  $t_1^{(1)}$  and  $t_2^{(1)}$  are real. One has  $P_1(1) = 2g_1 - 1 > 0$ , so, in view of (2.23), either  $0 < t_1^{(1)} \leq t_2^{(1)} < 1$  (c) or  $1 < t_1^{(1)} \leq t_2^{(1)}$  (d). One verifies with help of (2.19) that (c) is equivalent to  $L_1 = 1 - g_1 - 2g_1(1 - g_2) < 0$  and (d) to  $L_2 = 1 - g_1 - 2g_1(1 - g_2) > 0$ . Thus, in view of (2.17), if (d) holds, the equation  $(A_1)$  will have solution  $x = -1 + t_1^{(1)2} \in (0, +\infty)$  and if (c) holds,  $(\tilde{B}_1)$  will have solution  $x = -1 + t_2^{(1)2} \in (-1, 0)$  in view of (2.17).

As in proof of Theorem 2.1, since  $\eta(z)$  is a bijection of  $(-K, K)$  to  $(-1, +\infty)$ , there exists unique real number  $\tilde{T} \in (-K, K)$  satisfying equation  $\eta(\tilde{T}) = x$  with  $x \in (-1, +\infty)$  defined above.



Let

$$\zeta = \frac{T\tilde{T}}{K} \in (-T, T). \quad (2.24)$$

Then, as follows from (2.2), (2.3), there exists  $\tau$  satisfying (2.16). One has  $\tau \in (0, \zeta)$  if  $\zeta > 0$  and  $\tau \in (\zeta, 0)$  if  $\zeta < 0$ . That finishes the proof.  $\square$

The next corollary contains explicit formulas and precises the intervals containing the point of return  $\tau$  defined in Theorem 2.2.

**Corollary 2.1.** *We assume all conditions of Theorem 2.2 being satisfied and define four following subsets of  $(0, 1)^2$  (see Fig. 3):*

$$E = \{(g_1, g_2) \in (0, 1)^2 : D_1 \geq 0, 0 < g_1 < 1/2\}, \quad (2.25)$$

$$F = \{(g_1, g_2) \in (0, 1)^2 : D_1 \geq 0, 1/2 < g_1 < 1\}, \quad (2.26)$$

$$G = \{(g_1, g_2) \in (0, 1)^2 : D_2 \geq 0, 0 < g_1 < 1/2\}, \quad (2.27)$$

$$H = \{(g_1, g_2) \in (0, 1)^2 : D_2 \geq 0, 1/2 < g_1 < 1\}. \quad (2.28)$$

Let  $t_j^{(i)}$ ,  $i, j = 1, 2$  be defined by (2.19), (2.21) and  $\eta^{-1}(z)$  by (2.12), then

$$\zeta = \frac{T}{K} \eta^{-1}(-1 + t_1^{(2)2}) \in (0, +\infty) \quad \text{if} \quad (g_1, g_2) \in E \quad (2.29)$$

$$\zeta = \frac{T}{K} \eta^{-1}(-1 + t_2^{(2)2}) \in (-T, 0) \quad \text{if} \quad (g_1, g_2) \in F \quad (2.30)$$

$$\zeta = \frac{T}{K} \eta^{-1}(-1 + t_2^{(1)2}) \in (-T, 0) \quad \text{if} \quad (g_1, g_2) \in G \quad (2.31)$$

$$\zeta = \frac{T}{K} \eta^{-1}(-1 + t_1^{(1)2}) \in (0, +\infty) \quad \text{if} \quad (g_1, g_2) \in H \quad (2.32)$$

For  $\tau$  defined by (2.16) one has

$$\tau \in (0, \zeta) \quad \text{if} \quad \zeta > 0 \quad \text{and} \quad \tau \in (\zeta, 0) \quad \text{if} \quad \zeta < 0. \quad (2.33)$$

One easily verifies using the above formulas that  $\zeta(g_1, g_2) \rightarrow 0$  as  $g_1 \rightarrow 1/2$ .

### 3. FUNCTIONS BOUNDED IN THE COMPLEX STRIP

We denote  $\mathbb{A}_{M, \infty, B}$  the set of functions  $f(z)$  satisfying the following conditions

1.  $f(z)$  is holomorphic in the infinite strip  $S_B = \{z \in \mathbb{C} : |\operatorname{Im}(z)| < B\}$ ,  $B > 0$ .
2.  $f(\mathbb{R}) \subset \mathbb{R}$ .
3.  $|f(z)| < M$ ,  $M > 0$ ,  $\forall z \in S_B$ .

The next result gives characterization of functions bounded in absolute value in  $S_B$  with help of g-fractions and can be considered as the limit case of (1.12) as  $T \rightarrow +\infty$ .

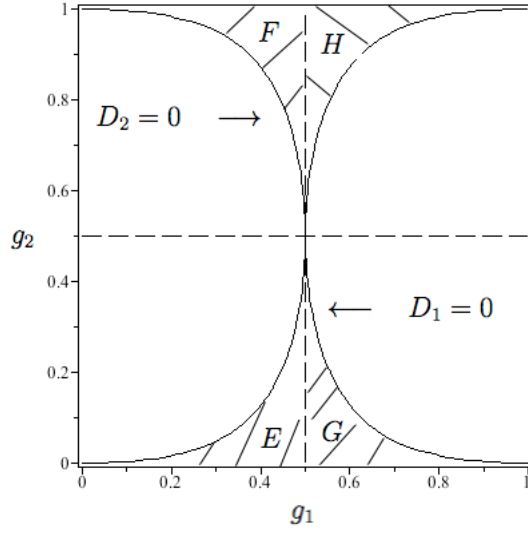


FIGURE 3. Four domains  $E, F, G, H$  in the parameter space  $(g_1, g_2) \in (0, 1)^2$  corresponding to the oscillatory behaviour of  $f(z) \in \mathbb{A}_{M,T,B}$ .

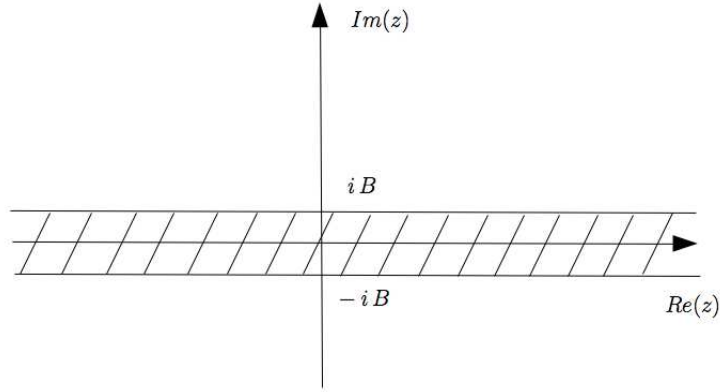


FIGURE 4. Domain of analyticity  $S_B$  of  $f(z) \in \mathbb{A}_{M,\infty,B}$ .

**Theorem 3.1.** *Let  $f(z) \in \mathbb{A}_{M,\infty,B}$ . Then for some  $\mu_0 > 0$  and  $g_k \in [0, 1]$ ,  $k \geq 1$  one has*

$$f(z) = M \left( 1 - \frac{2}{\mu_0 \exp\left(\frac{\pi z}{2B}\right) \{g_1, g_2, \dots | \exp\left(\frac{\pi z}{B}\right) - 1\} + 1} \right). \quad (3.1)$$

*Proof.* Let  $m : \mathbb{D}_M \rightarrow \mathbb{H}_+$  be defined by (1.7). We define the conformal map

$$l(z) = \frac{B}{\pi} \log(1 + z), \quad l : \mathbb{H} \rightarrow S_B. \quad (3.2)$$

One verifies that the composition  $F = m \circ f \circ l$  is holomorphic in  $\mathbb{H}$  and  $F(\mathbb{H}) \subset \mathbb{H}_+$  with  $F(z) \in \mathbb{R}$  for  $z > -1$ . Thus, according to theorem of Wall [8], p. 279  $F$  can be written as follows

$$F(z) = \mu_0 \sqrt{1 + z} \int_0^1 \frac{d\mu(u)}{1 + zu}, \quad (3.3)$$

for some nondecreasing real bounded function  $\mu(u)$ ,  $u \in (0, 1)$  and  $\mu_0 > 0$ .

For  $f = m^{-1} \circ F \circ l^{-1}$  one obtains the following formula

$$f(z) = M \left( 1 - \frac{2}{\mu_0 \exp\left(\frac{\pi z}{2B}\right) \int_0^1 \frac{d\mu(u)}{1 + (\exp\left(\frac{\pi z}{B}\right) - 1)u} + 1} \right). \quad (3.4)$$

The integral in (3.4) can be transformed to the continued  $g$ -fraction form [8]

$$\int_0^1 \frac{d\mu(u)}{1 + (\exp\left(\frac{\pi z}{B}\right) - 1)u} = \left\{ g_1, g_2, \dots \mid \exp\left(\frac{\pi z}{B}\right) - 1 \right\}, \quad \text{for some } g_k \in [0, 1], \quad (3.5)$$

that together with (3.4) implies (3.1). Proof is finished.  $\square$

Let

$$\theta(z) = \frac{1}{M} f(Bz/\pi). \quad (3.6)$$

To calculate the coefficients  $g_p$  in (3.1) one has formulas similar to (1.27):

$$g_p = C_p(\theta(0), \theta'(0), \dots, \theta^{(p)}(0)), \quad p \geq 1, \quad (3.7)$$

with rational functions  $C_p$  determined by calculation of derivatives of both sides of (3.1) at  $z = 0$ .

Introducing

$$\theta_n = \theta^{(n)}(0) = \frac{1}{M} \frac{B^n}{\pi^n} f^{(n)}(0), \quad n \geq 0, \quad (3.8)$$

we provide below explicit formulas for  $\mu_0$ ,  $g_1$ ,  $g_2$ :

$$\mu_0 = \frac{1 + \theta_0}{1 - \theta_0}, \quad (3.9)$$

$$g_1 = \frac{1}{2} \frac{1 - 4\theta_1 - \theta_0^2}{1 - \theta_0^2}, \quad (3.10)$$

$$g_2 = \frac{1}{2} \frac{(16\theta_1^2 - 8\theta_2 - \theta_0 + \theta_0^2 + \theta_0^3 - 8\theta_2\theta_0 - 1)(1 - \theta_0)}{(1 - \theta_0^2 + 4\theta_1)(4\theta_1 - 1 + \theta_0^2)}. \quad (3.11)$$

**Definition 3.1.** We denote by  $\mathbb{A}_{M,\infty,B}^{(k)} \subset \mathbb{A}_{M,\infty,B}$ ,  $k = 1, 2, \dots$  the set of functions for which the  $g$ -fraction representation (1.12) satisfies the condition

$$g_i \notin \{0, 1\}, \quad \forall i = 1, \dots, k. \quad (3.12)$$

The next result is analogous to Theorem 2.1 and is proved in the similar way.

**Theorem 3.2.** *Let  $f(z) \in \mathbb{A}_{M,\infty,B}^{(1)}$ ,  $f'(0) \neq 0$  where  $g_1$  is defined by (3.8) and (3.10) as function of  $M, B, f(0), f'(0)$ . Let  $\tau \in \mathbb{R}$ ,  $\tau \neq 0$  be the point such that  $f(\tau) = f(0)$ . We define*

$$\tilde{\tau} = \frac{2B}{\pi} \left| \log \left( \frac{1 - g_1}{g_1} \right) \right| > 0. \quad (3.13)$$

*Then*

$$|\tau| \geq \tilde{\tau}. \quad (3.14)$$

The following theorem is equivalent to Theorem 2.2 for functions  $f \in \mathbb{A}_{M,\infty,B}^{(2)}$  and has the similar proof.

**Theorem 3.3.** *Let  $f(z) \in \mathbb{A}_{M,\infty,B}^{(2)}$ ,  $f'(0) \neq 0$  where  $g_1, g_2$  are defined by formulas (3.8) and (3.10), (3.11). We assume that the point  $(g_1, g_2) \in (0, 1)^2$  belongs to one of the four regions  $E, F, G, H$  defined by (2.25)-(2.28).*

*Let*

$$\zeta = \frac{2B}{\pi} \log(t_2^{(2)}) < 0 \quad \text{if} \quad (g_1, g_2) \in E \quad (3.15)$$

$$\zeta = \frac{2B}{\pi} \log(t_1^{(2)}) > 0 \quad \text{if} \quad (g_1, g_2) \in F \quad (3.16)$$

$$\zeta = \frac{2B}{\pi} \log(t_2^{(1)}) < 0 \quad \text{if} \quad (g_1, g_2) \in G \quad (3.17)$$

$$\zeta = \frac{2B}{\pi} \log(t_1^{(1)}) > 0 \quad \text{if} \quad (g_1, g_2) \in H \quad (3.18)$$

where  $t_j^{(i)}$ ,  $i, j = 1, 2$  are defined as functions of  $g_1, g_2$  by (2.19) and (2.21).

*Then there exists  $\tau \in (-T, T)$ ,  $\tau \neq 0$  such that*

$$f(\tau) = f(0). \quad (3.19)$$

*and  $\tau \in (0, \zeta)$  if  $\zeta > 0$  and  $\tau \in (\zeta, 0)$  if  $\zeta < 0$ .*

To conclude this section we will consider the case of functions analytic and bounded in a semi-infinite strip. Let  $B > 0, T > 0$ . We denote by  $S_{B,T} \subset \mathbb{C}$  the complex domain which is the interior of the semi-infinite strip formed by segments  $t + iB, t - iB, t \in [-T, +\infty)$  and  $-T + it, t \in [-B, B]$  (see Fig. 5).

Let  $\mathbb{B}_{M,T,B}$  be the set of functions  $f(z)$  satisfying the following conditions:

1.  $f(z)$  is holomorphic in  $S_{B,T}$ .
2.  $f(\mathbb{R}) \subset \mathbb{R}$ .
3.  $|f(z)| < M, M > 0, \forall z \in S_{B,T}$ .

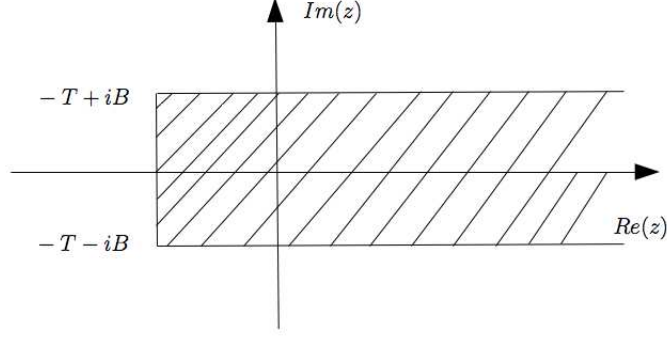


FIGURE 5. Domain of analiticity  $S_{T,B}$  of  $f(z) \in \mathbb{D}_{M,T,B}$ .

It is straighforward to verify that the following function

$$L(z) = \frac{\cosh\left(\frac{\pi(z+T)}{B}\right) - \cosh\left(\frac{\pi T}{B}\right)}{\cosh\left(\frac{\pi T}{B}\right) - 1}, \quad (3.20)$$

defines a conformal map  $L : S_{B,T} \rightarrow \mathbb{H}$  and maps bijectively  $(-T, +\infty)$  to  $(-1, +\infty)$ ,  $L(0) = 0$ . One can formulate now the result analogous to (1.12) and (3.1):

**Theorem 3.4.** *Let  $f(z) \in \mathbb{B}_{M,T,B}$ . Then for some  $\mu_0 > 0$  and  $g_k \in [0, 1]$ ,  $k \geq 1$  one has*

$$f(z) = M \left( 1 - \frac{2}{\mu_0 \sqrt{1 + L(z)} \{g_1, g_2, \dots | L(z)\} + 1} \right). \quad (3.21)$$

#### 4. APPLICATIONS

We will apply the results from previous sections to the Newtonian three-body problem, whose solutions in many situations are analytic functions in the strip along the real axis of the complex time plane.

We consider three mass points  $P_1, P_2, P_3$  in  $\mathbb{R}^3$  which attract each other according to the Newtonian law with finite positive masses  $m_1, m_2, m_3$ . Let  $R_i = (x_i, y_i, z_i)$  be the position vector of  $P_i$  and  $r_{ij}$  the distance between it and mass  $j$ . One writes equations of motion as follows:

$$m_i \frac{dR'_i}{dz} = - \sum_{j \neq i} m_i m_j \frac{R_i - R_j}{r_{ij}^3}, \quad (4.1)$$

$$R'_i = \frac{dR'_i}{dz} = (x'_i, y'_i, z'_i), \quad i = 1, 2, 3, \quad (4.2)$$

which have the integral of energy:

$$H = T + U = h = -\frac{m_1 m_2 m_3}{2\Gamma} K, \quad h, K = \text{const}, \quad (4.3)$$

$$T = \sum_{i=1}^3 \frac{x_i'^2 + y_i'^2 + z_i'^2}{2}, \quad (4.4)$$

$$U = -\frac{m_3 m_2}{r_{32}} - \frac{m_1 m_3}{r_{13}} - \frac{m_2 m_1}{r_{21}}, \quad (4.5)$$

$$\Gamma = m_1 + m_2 + m_3, \quad (4.6)$$

the first integrals of the impulse of the system:

$$\sum_{i=1}^3 m_i x_i = 0, \quad \sum_{i=1}^3 m_i z_i = 0, \quad \sum_{i=1}^3 m_i z_i = 0, \quad (4.7)$$

$$\sum_{i=1}^3 m_i x'_i = 0, \quad \sum_{i=1}^3 m_i y'_i = 0, \quad \sum_{i=1}^3 m_i z'_i = 0, \quad (4.8)$$

and the first integrals of the angular momentum:

$$\sum_{i=1}^3 m_i (x_i y'_i - x'_i y_i) = c_1, \quad \sum_{i=1}^3 m_i (y_i z'_i - y'_i z_i) = c_2, \quad \sum_{i=1}^3 m_i (z_i x'_i - z'_i x_i) = c_3, \quad (4.9)$$

$c_1, c_2, c_3 = \text{const.}$

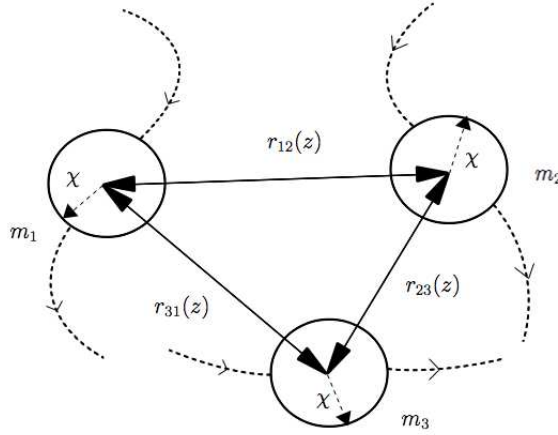


FIGURE 6. The planar three-planet problem.

We shall need the following result due to Sundman [3].

**Theorem 4.1.** *Let  $J \subset \mathbb{R}$  be a connected open interval and  $x_i(z), y_i(z), z_i(z), i = 1, 2, 3$  be a solution of the three-body problem (4.1) defined for  $z \in J$  and satisfying the following inequalities*

$$r_{32}(z) > \chi, \quad r_{13}(z) > \chi, \quad r_{21}(z) > \chi, \quad \forall z \in J. \quad (4.10)$$

*Then  $\forall z_0 \in J$  the positions  $R_i(z), i = 1, 2, 3$  are holomorphic functions in the complex disk*

$$\Delta_\chi = \{z \in \mathbb{C} : |z - z_0| < B_\chi\}, \quad (4.11)$$

where

$$B_\chi = \frac{1}{14} \frac{\chi}{\sqrt{\frac{56}{21} \frac{\Gamma^2}{m_\chi} + \Gamma|K|}}, \quad m = \min\{m_1, m_2, m_3\}, \quad (4.12)$$

and satisfy  $\forall z \in \Delta_\chi$  the inequalities

$$|x_i(z_0) - x_i(z)| < \chi/14, \quad |y_i(z_0) - y_i(z)| < \chi/14, \quad |z_i(z_0) - z_i(z)| < \chi/14, \quad i = 1, 2, 3. \quad (4.13)$$

Such a solution can be interpreted as a collision free motion for  $z \in J$  of three planets each of radius  $\chi > 0$  ( see Fig. 6).

**Lemma 4.1.** *Let all conditions of Theorem 4.1 hold and  $z_0 \in J$ . Then the inverse mutual distances  $r_{ij}^{-1}(z)$  are holomorphic functions in  $\Delta_\chi$  and bounded in the absolute value:*

$$|r_{ij}(z)^{-1}| < M_\chi = \frac{7}{2\chi}, \quad \forall z \in \Delta_\chi. \quad (4.14)$$

*Proof.* Let  $\langle, \rangle$  denotes the Euclidean scalar product in  $\mathbb{R}^3$ . For some fixed  $i < j$  we introduce  $X(z) = (X_1(z), X_2(z), X_3(z)) = r_i(z)$ ,  $Y(z) = (Y_1(z), Y_2(z), Y_3(z)) = r_j(z)$ . Let  $\tilde{X}(z) = X(z_0) + \tilde{X}(z)$ ,  $\tilde{Y}(z) = Y(z_0) + \tilde{Y}(z)$ ,  $z \in \Delta_\chi$  where  $\tilde{X}(z) = (\tilde{X}_1(z), \tilde{X}_2(z), \tilde{X}_3(z))$ ,  $\tilde{Y}(z) = (\tilde{Y}_1(z), \tilde{Y}_2(z), \tilde{Y}_3(z))$ . Then

$$|\tilde{X}_i(z)| < \chi/14, \quad |\tilde{Y}_i(z)| < \chi/14, \quad \forall z \in \Delta_\chi, \quad i = 1, 2, 3, \quad (4.15)$$

as follows from Sundman's Theorem 4.1.

Let  $E = X(z_0) - Y(z_0)$ ,  $R^2 = \langle E, E \rangle$ ,  $K = \tilde{X}(z) - \tilde{Y}(z)$ . Then  $|r_{ij}^2(z)| = |\langle E + K, E + K \rangle|$ . Applying the triangular inequality one obtains for  $z \in \Delta_\chi$ :

$$|r_{ij}^2(z)| \geq R^2 - 2|\langle E, K \rangle| - |\langle K, K \rangle| > R^2 - 12R\frac{\chi}{14} - 12\left(\frac{\chi}{14}\right)^2 = \left(\frac{2\chi}{7}\right)^2, \quad (4.16)$$

where we have used the inequalities  $R^2 > \chi^2$  and (4.15). That implies (4.14) and finishes the proof.  $\square$

We shall consider the case then  $J = \mathbb{R}$  which corresponds to the collision-free motion of three rigid spherical bodies, each of radius  $\chi > 0$ , for  $-\infty < z < +\infty$ . In this case, according to Sundman's Theorem 4.1, all inverse mutual distances  $r_{ij}^{-1}(z)$  are analytic functions in the complex infinite strip  $S_{B_\chi}$ . As follows from Lemma 4.1,  $r_{ij}^{-1}(z)$  are bounded in  $S_{B_\chi}$  in absolute value by  $M_\chi$ . Using the equations of motion (4.1) it is easy to proof that  $r_{ij}^{-1}(z) \in A_{M, \infty, B}^{(k)}$ ,  $k = 1, 2$ . Thus, the Theorems 3.2 and 3.3 can be applied in this case. Below we give describe one possible application to the collision problem:

*Let us consider the motion of three spherical rigid bodies (planets) in 3-dimensional Euclidean space each of radius  $\chi > 0$  with masses  $m_1, m_2, m_3$  interacting according to Newtonian law. The motion is assumed to be collision free inside some small interval  $z \in (-\epsilon, \epsilon)$ ,  $\epsilon > 0$ . Let  $r_{ij}(z) \in \{r_{32}(z), r_{13}(z), r_{21}(z)\}$  be one of three mutual distances.*

We put  $f(z) = r_{ij}^{-1}(z)$ . One calculates  $f(0), f'(0), f''(0)$  and finds the upper and lower bounds on the time of the first return of  $f(t)$  to its initial value  $f(0)$  using Theorems 3.2 and 3.3 with  $B = B_\chi$ ,  $M = M_\chi$  defined by (4.12), (4.14). Then, if these bounds are not satisfied for the observed motion, then one concludes that  $r_{ij}^{-1}(z) \notin A_{M,\infty,B}^{(k)}$ ,  $k = 1, 2$  and thus there has to be a collision between bodies for some negative or positive value of time  $z$ .

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